Single integro-differential wave equation for Lévy walk

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The integro-differential wave equation for the probability density function for a classical one-dimensional Lévy walk with continuous sample paths has been derived. This equation involves a classical wave operator together with memory integrals describing the spatio-temporal coupling of the Lévy walk. It is valid for any running time PDF of a walker and it does not involve any long-time large-scale approximations. It generalizes the well-known telegraph equation obtained from the persistent random walk. Several non-Markovian cases are considered when the particle's velocity alternates at the gamma and power-law distributed random times.

PACS numbers:

Lévy walk is a fundamental notion in physics and biology [1] with numerous applications including the transport of light governed by Lévy statistics [2], anomalous superdiffusion of cold atoms in optical lattices [3], T-cell motility in the brain [4], endosomal active transport within living cells [5]. In the last few years, the interest in the Lévy walk models increases rapidly in the context of Lévy flight foraging hypothesis according to which a Lévy transport is optimal to search for randomly located objects [6]. Lévy movement pattern has been observed in microorganisms, insects, molluscs, birds, etc. [7]. Recent publication *Lévy walk* in *Review of Modern Physics* provides a detailed discussion of existing applications and the most current status of Lévy walk theory [8].

The derivation of the governing equations for a Lévy walk from an underlying stochastic movement is a long-standing problem. Many contributions on this subject have been given since the pioneering works three decades ago (see, for example, [9, 10]). The standard model for a Lévy walk is based on the continuos time random walk (CTRW) with coupled probability density function for the jump length and the waiting time between two successive jumps. Two integral equations for the probability density function (PDF) $p(x,t)$ for a walker position x at time t and arrival rate $j(x,t)$ can be formulated and solved by the Fourier-Laplace transform technique [1, 8]. An equivalent single integral equation for $p(x,t)$ has been also formulated [11]. To describe a Lévy walk spatiotemporal coupling, two dynamical equations involving fractional material derivatives have been suggested in [12]. Another approach to describe the superdiffusive behavior is based on the analysis of joint PDF $p(x,v,t)$ of the particle's position x and velocity v. Various fractional generalizations of Kramers-Fokker-Planck equation for $p(x,v,t)$ have been derived [13–15]. However, to the author's knowledge, the single integro-differential equation for the Lévy walk PDF $p(x,t)$ is still not available in the literature. In this paper we obtain the following integro-differential wave equation

$$\frac{\partial^2 p}{\partial t^2} - v^2 \frac{\partial^2 p}{\partial x^2} + \int_0^t \int_V K(\tau)\,\varphi(u) \left(\frac{\partial}{\partial t} - u \frac{\partial}{\partial x} \right) \times p(x-u\tau, t-\tau)\, du d\tau = 0, \qquad (1)$$

where v is a constant speed of walker, $\varphi(u)$ is the velocity jump density:

$$\varphi(u) = \frac{1}{2}\delta(u-v) + \frac{1}{2}\delta(u+v) \qquad (2)$$

in the velocity space V and $K(\tau)$ is the standard memory kernel from the CTRW theory [1]. It is determined by its Laplace transform $\hat{K}(s) = \hat{\psi}(s)/\hat{\Psi}(s)$, where $\hat{\psi}(s)$ and $\hat{\Psi}(s)$ are the Laplace transforms of the running time density $\psi(\tau)$ and the survival function $\Psi(\tau)$. From Eq. (1) with $p(x,0) = p_0(x)$ and $p_t(x,0) = 0$, one can obtain the well-known expression for the Fourier-Laplace transform of the PDF $p(x,t)$

$$\hat{p}(k,s) = \frac{\left[\hat{\Psi}(s+ikv) + \hat{\Psi}(s-ikv)\right]\hat{p}_0(k)}{2 - \hat{\psi}(s+ikv) - \hat{\psi}(s-ikv)}, \qquad (3)$$

where $\hat{p}_0(k) = \int_{\mathbb{R}} p_0(x)\, e^{ikx} dx$ [8].

Derivation. We consider the Lévy walk as the random particle's motion with continuos sample paths (no jumps) along one-dimensional space. The particle starts to move with constant speed v at time $t=0$ and after a random time (running time) it either continues the movement in the same direction with curtain probability or changes the direction and moves with the same constant speed. The random running time is defined by the switching rate $\gamma(\tau)$ or the running time PDF $\psi(\tau) = \gamma(\tau)\exp\left[-\int_0^\tau \gamma(s)ds\right]$. To derive the governing equation for the Lévy walk, we start with Markovian model involving structural densities with the extra running time variable τ [16–20]. We define the structural PDF's of walker, $n_+(x,t,\tau)$, at point x and time t that moves in the right direction, $(+)$, with constant speed v during time τ since the last switching. The probability density function $n_-(x,t,\tau)$ corresponds to the walker that moves in the negative direction, $(-)$. The balance equations for both structural PDF's $n_+(x,t,\tau)$ and $n_-(x,t,\tau)$ can be written as

$$\frac{\partial n_\pm}{\partial t} \pm v \frac{\partial n_\pm}{\partial x} + \frac{\partial n_\pm}{\partial \tau} = -\gamma(\tau) n_\pm, \qquad (4)$$

where the switching rate $\gamma(\tau)$ depends on the running time τ. If the walker moves in the positive direction it

can switch with rate $\gamma(\tau)$ either to the opposite direction with the probability α_- or keep the same direction with the probability α_+ such that $\alpha_+ + \alpha_- = 1$. For the walker moving in the negative direction corresponding characteristics are β_+ and β_-. The well-known *velocity model* and *two-state model* are just particular cases of this general two-state model. For example, the choice $\alpha_+ = \beta_- = 0$ and $\alpha_- = \beta_+ = 1$ corresponds to the two-state model [21]. The probabilities $\alpha_\pm = \beta_\pm = 1/2$ correspond to the velocity model [22]. We assume that at the initial time $t = 0$ all walkers start to move with zero running time τ

$$n_\pm(x, 0, \tau) = \frac{1}{2} p_0(x)\delta(\tau). \tag{5}$$

Here we consider the symmetrical initial conditions when walker start to move on the right with probability 1/2 and on the left with the same probability. The boundary conditions at zero running time can be formulated as

$$\begin{aligned} n_\pm(x, t, 0) &= \alpha_\pm \int_0^t \gamma(\tau) n_+(x, t, \tau) d\tau + \\ &\quad \beta_\pm \int_0^t \gamma(\tau) n_-(x, t, \tau) d\tau. \end{aligned} \tag{6}$$

Our aim is to derive the master equations for the probability density functions $p_+(x, t)$ and $p_-(x, t)$ defined as

$$p_\pm(x, t) = \int_0^t n_\pm(x, t, \tau) d\tau. \tag{7}$$

By differentiating (7) with respect to time t and using the balance equations (4) we obtain

$$\begin{aligned} \frac{\partial p_\pm}{\partial t} &= n_\pm(x, t, t) \mp v \int_0^t \frac{\partial n_\pm}{\partial x} d\tau \\ &\quad - \int_0^t \frac{\partial n_\pm}{\partial \tau} d\tau - \int_0^t \gamma(\tau) n_\pm(x, t, \tau) d\tau. \end{aligned}$$

This equation can be rewritten as the master equation

$$\frac{\partial p_\pm}{\partial t} \pm v \frac{\partial p_\pm}{\partial x} = j_\pm(x, t) - i_\pm(x, t), \tag{8}$$

where the rates of switching $i_\pm(x, t)$ are defined as

$$i_\pm(x, t) = \int_0^t \gamma(\tau) n_\pm(x, t, \tau) d\tau \tag{9}$$

and the arrival rates $j_\pm(x, t)$ are

$$j_\pm(x, t) = n_\pm(x, t, 0). \tag{10}$$

By using the definitions of switching and arrival rates (9) and (10), Eq. (6) can be written in the compact form

$$j_\pm(x, t) = \alpha_\pm i_+(x, t) + \beta_\pm i_-(x, t). \tag{11}$$

In what follows we consider only the simple case of a symmetric Lévy walk for which $\alpha_\pm = \beta_\pm = 1/2$. In general, the probabilities $\alpha_\pm$ and $\beta_\pm$ can be useful to formulate the impact of the external force or chemotactic substance. Substitution of (11) with $\alpha_\pm = \beta_\pm = 1/2$ into the master equation (8) gives

$$\frac{\partial p_+}{\partial t} + v \frac{\partial p_+}{\partial x} = -\frac{1}{2} i_+(x, t) + \frac{1}{2} i_-(x, t), \tag{12}$$

$$\frac{\partial p_-}{\partial t} - v \frac{\partial p_-}{\partial x} = \frac{1}{2} i_+(x, t) - \frac{1}{2} i_-(x, t), \tag{13}$$

where the switching rates $i_\pm(x, t)$ can be found as follows. By the method of characteristics, we find from (4)

$$n_\pm(x, t, \tau) = n_\pm(x \mp v\tau, t - \tau, 0)\Psi(\tau), \quad \tau < t, \tag{14}$$

where $\Psi(\tau)$ is the survival function

$$\Psi(\tau) = e^{-\int_0^\tau \gamma(s) ds}. \tag{15}$$

Note that at $\tau = t$ we have a singularity due to the initial condition (5). Substitution of (14) into (9) and (7) together with the initial condition (5) gives

$$i_\pm(x, t) = \int_0^t j_\pm(x \mp v\tau, t - \tau)\psi(\tau) d\tau + \frac{1}{2} p_0(x \mp vt)\psi(t),$$

$$p_\pm(x, t) = \int_0^t j_\pm(x \mp v\tau, t - \tau)\Psi(\tau)\, d\tau + \frac{1}{2} p_0(x \mp vt)\Psi(t).$$

Applying the Fourier-Laplace transform to these equations, we can find expressions for $i_+(x, t)$ and $i_-(x, t)$ in terms of $\rho_+(x, t)$ and $\rho_-(x, t)$ as [20]

$$i_\pm(x, t) = \int_0^t K(\tau) p_\pm(x \mp v\tau, t - \tau) d\tau. \tag{16}$$

Note that four integral equations $i_\pm(x, t)$ and $p_\pm(x, t)$ above can be written in the standard form of two equations for the PDF:

$$p(x, t) = p_+(x, t) + p_-(x, t). \tag{17}$$

and the arrival rate $j = j_+ + j_-$ with the jump density $w(z|\tau) = \frac{1}{2}\delta(z - v\tau) + \frac{1}{2}\delta(z + v\tau)$. For the symmetrical case $j = i_+ + i_-$ we obtain the well-known equations

$$\begin{aligned} p(x, t) &= \int_0^t \int_{\mathbb{R}} j(x - z, t - \tau) w(z|\tau) \Psi(\tau)\, dz d\tau + \\ &\quad \int_{\mathbb{R}} p_0(x - z) w(z|t) dz \Psi(t), \end{aligned}$$

$$\begin{aligned} j(x, t) &= \int_0^t \int_{\mathbb{R}} j(x - z, t - \tau) w(z|\tau) \psi(\tau) dz d\tau + \\ &\quad \int_{\mathbb{R}} p_0(x - z) w(z|t) dz \psi(t). \end{aligned}$$

Note that these two equations are the starting point for

the various studies of the Lévy walk [8].

Our main purpose is to reduce the system (12), (13) with (16) to a single governing equation for the PDF $p(x,t)$ defined by (17). First we introduce the flux [24]

$$J(x,t) = vp_+(x,t) - vp_-(x,t). \tag{18}$$

Then by adding (12) and (13) we obtain the standard conservation equation

$$\frac{\partial p}{\partial t} + \frac{\partial J}{\partial x} = 0. \tag{19}$$

Equation for the flux J can be obtain by multiplication of (12) and (13) by v and subtraction

$$\frac{\partial J}{\partial t} + v^2 \frac{\partial p}{\partial x} = -v\left[i_+(x,t) - i_-(x,t)\right]. \tag{20}$$

By differentiating (19) with respect to t and (20) with respect to x and eliminating $\partial^2 J/\partial t \partial x$, we obtain

$$\frac{\partial^2 p}{\partial t^2} = v^2 \frac{\partial^2 p}{\partial x^2} + v\frac{\partial}{\partial x}\left[i_+(x,t) - i_-(x,t)\right]. \tag{21}$$

Now we need to express the last term in (21) in terms of $p(x,t)$ alone. From (17) and (18) we find the expressions for $p_+(x,t)$ and $p_-(x,t)$ in terms of the PDF $p(x,t)$ and the flux $J(x,t)$

$$p_\pm(x,t) = \frac{p(x,t)}{2} \pm \frac{J(x,t)}{2v}. \tag{22}$$

Substitution of (22) into (16) gives

$$v\frac{\partial}{\partial x}\left[i_+(x,t) - i_-(x,t)\right] =$$

$$\frac{1}{2}\int_0^t K(\tau)\,[\frac{\partial J}{\partial x}(x - v\tau, t-\tau) + \frac{\partial J}{\partial x}(x + v\tau, t-\tau)$$
$$+v\frac{\partial p}{\partial x}(x - v\tau, t-\tau) - v\frac{\partial p}{\partial x}(x + v\tau, t-\tau)]d\tau. \tag{23}$$

By using (23) together with (19) and (21), we obtain a single integro-differential equation for the PDF $p(x,t)$

$$\frac{\partial^2 p}{\partial t^2} = v^2 \frac{\partial^2 p}{\partial x^2} -$$

$$\frac{1}{2}\int_0^t K(\tau)\left[\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)p\,(x - v\tau, t-\tau)\right]d\tau -$$
$$\frac{1}{2}\int_0^t K(\tau)\left[\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)p\,(x + v\tau, t-\tau)\right]d\tau. \tag{24}$$

This integro-differential wave equation for a *Lévy walk* is valid for any running time PDF $\psi(\tau)$. It generalizes the well-known telegraph equation obtained from the persistent random walk with the constant rate of switching γ. By using the velocity jump density (2), we rewrite Eq. (24) in the compact form (1). It is instructive to compare the uncoupled CTRW with Lévy walk. When jumps and waiting times of the CTRW are independent random variables, one can convert the single integral equation for $p(x,t)$ into a master equation [1, 16]. For the Lévy walk involving spatio-temporal coupling the integro-differential wave equation (24) plays the same role as the master equation for the uncoupled CTRW.

Let us now consider several examples of the running time PDF $\psi(\tau)$.

Exponential running time density. In the Markovian case with the exponential running time PDF

$$\psi(\tau) = \frac{1}{T}\exp\left(-\frac{\tau}{T}\right) \tag{25}$$

for which $\hat{\psi}(s) = (1+Ts)^{-1}$ and $K(\tau) = T^{-1}\delta(\tau)$, we obtain from (24) the classical Cattaneo or telegraph equation [16]

$$\frac{\partial^2 p}{\partial t^2} + \frac{1}{T}\frac{\partial p}{\partial t} - v^2\frac{\partial^2 p}{\partial x^2} = 0. \tag{26}$$

This hyperbolic equation ensures that the density profile propagates with finite speed v.

Gamma PDF $g(\tau, 2, \lambda)$. For the biological applications it is important to consider a running time PDF that takes the maximum value not at zero time as the exponential density (25) [7]. For example, it was found that the running time density for a single bacterium might deviate significantly from exponential approximation [23]. One example of such PDF is the *gamma density*

$$\psi(\tau) = g(\tau, 2, \lambda) = \lambda^2 \tau \exp(-\lambda\tau) \tag{27}$$

with $\hat{\psi}(s) = \lambda^2\ (s+\lambda)^{-2}$ and $\hat{K}(s) = \lambda^2\ (2\lambda + s)^{-1}$. The memory kernel in Eq. (24) takes the form

$$K(\tau) = \lambda^2 \exp(-2\lambda\tau). \tag{28}$$

The advantage of this exponential memory kernel is that one can localize integro-differential equation (24) by direct differentiation of (24) with respect to time twice:

$$\frac{\partial^4 p}{\partial t^4} + 4\lambda\frac{\partial^3 p}{\partial t^3} + 5\lambda^2\frac{\partial^2 p}{\partial t^2} + 2\lambda^3\frac{\partial p}{\partial t} =$$

$$v^2\frac{\partial^2}{\partial x^2}\left[2\frac{\partial^2 p}{\partial t^2} + 4\lambda\frac{\partial p}{\partial t} + 3\lambda^2 p - v^2\frac{\partial^2 p}{\partial x^2}\right]. \tag{29}$$

Note that non-Markovian particle's movement with velocities alternating at Erlang-distributed and gamma-distributed random times was considered in [25, 26]. Next we consider the anomalous case involving walker's velocities alternating at power-law distributed random times [12, 19, 21].

Anomalous enhanced transport. We consider two anomalous cases: (1) strong ballistic case for which the mean squared displacement: $< x^2 > \sim t^2$ and (2) subballistic superdiffusion with $< x^2 > \sim t^{3-\mu}$, where $1 < \mu < 2$

[8]. In the ballistic case, we use the survival function $\Psi(\tau) = E[-(\tau/\tau_0)^{\mu}]$ with $0 < \mu < 1$ for which the mean running time is divergent [27, 28]. In this case

$$\hat{\psi}(s) = \frac{1}{1 + (\tau_0 s)^{\mu}}, \qquad 0 < \mu < 1 \tag{30}$$

and the Laplace transform of the memory kernel $K(\tau)$ is

$$\hat{K}(s) = \frac{s\hat{\psi}(s)}{1 - \hat{\psi}(s)} = \frac{s^{1-\mu}}{\tau_0^{\mu}}.$$

For this kernel the main equation (1) can be rewritten in the different forms by using material fractional derivatives [12, 29–33]. We write it in the form

$$\frac{\partial^2 p}{\partial t^2} - v^2 \frac{\partial^2 p}{\partial x^2} + \frac{1}{\tau_0^{\mu}} \int_V \varphi(u) L_u^{1-\mu} p du = 0, \tag{31}$$

where the operator $L_u^{1-\mu}$ is defined by its Fourier-Laplace transform

$$\mathcal{FL}\left\{L_u^{1-\mu} p\right\} = (s - iku)^{1-\mu} \left[(s + iku)\, p(k,s) - p_0(k)\right]. \tag{32}$$

In the subballistic superdiffusive case, one can obtain for small s

$$\hat{\psi}(s) \simeq 1 - Ts + ATs^{\mu}, \qquad 1 < \mu < 2 \tag{33}$$

for which the first moment $T = \int_0^{\infty} \Psi(\tau) d\tau$ is finite and the second moment is divergent. Then

$$\hat{K}(s) \simeq \frac{1}{T}\left(1 + As^{\mu-1}\right)$$

as $s \to 0$. From (1) we obtain the following equation

$$T\frac{\partial^2 p}{\partial t^2} + \frac{\partial p}{\partial t} - D\frac{\partial^2 p}{\partial x^2} + A \int_V \varphi(u) L_u^{1-\mu} p du = 0 \tag{34}$$

with the diffusion coefficient $D = Tv^2$. It is clear from (31) and (34) that ballistic and subballistic cases are fundamentally different. In the strong ballistic case $(0 < \mu < 1)$, the integral term is in the balance with classical wave equation, while for the subballistic superdiffusive case $(1 < \mu < 2)$, the memory term is in the balance with the Cattaneo (telegraph) equation. One can perform various asymptotic analysis of (31) and (34), obtain the pseudo-differential equations for the walker's PDF position [30–33] and determine the shape of PDF profiles [34].

As an illustration let us find the long-time asymptotic solution to Eq. (31). In the limit $\tau_0 \to 0$, the evolution of the PDF $p(x,t)$ is determined by the integral term, while the first two wave equation terms in (31) are irrelevant. The PDF $p(x,t)$ obeys

$$\int_V \varphi(u) L_{1-\mu}^{u} p du = 0. \tag{35}$$

By using (32), we obtain

$$(s - ikv)^{1-\mu} \left[(s + ikv)\, \hat{p}(k,s) - \hat{p}_0(k)\right] +$$

$$(s + ikv)^{1-\mu} \left[(s - ikv)\, \hat{p}(k,s) - \hat{p}_0(k)\right] = 0. \tag{36}$$

In particular, for $\mu = 1/2$ and $\hat{p}_0(k) = 1$, we factorize Eq. (36) and find

$$\left(s^2 + k^2 v^2\right)^{\frac{1}{2}} \hat{p}(k,s) - 1 = 0.$$

By using the inverse Fourier-Laplace transform we obtain the well-known self-similar profile [1]

$$p(x,t) = \pi^{-1} \left(v^2 t^2 - x^2\right)^{-\frac{1}{2}}.$$

For the arbitrary μ from the interval $0 < \mu < 1$, one can find from (36) that the solution to (35) is the Lamperti distribution [8]. It would be interesting to use a new wave equation for the non-normalizable density problem for superdiffusive anomalous transport [35]. Note that our equation can be generalized for the case when the particle moves with the random velocity: $\dot{x}_{\pm}(t) = \pm v + \dot{B}(t)$, where $\dot{B}(t)$ is the Gaussian white noise.

In summary, we derived the integro-differential wave equation for the probability density function for a position of a Lévy walker with continuous sample paths. This equation involves a classical wave operator together with memory integrals describing a spatio-temporal coupling of Lévy walk. It is valid for any running time PDF and it does not involve any long-time large-scale approximations. It generalizes the well-known telegraph equation obtained from the persistent random walk. For the Lévy walk the integro-differential wave equation (1) plays the same role as the master equation for the uncoupled CTRW [1, 16]. Our technique may lead to the significant advances in the extension of the linear Lévy walk models. Our approach may be particularly helpful when we deal with the *Lévy particles* interactions [36] when it would be difficult to take into account nonlinear terms within the standard approach involving integral equations (see the similar problems for subdiffusion [37, 38]). The description of the *Lévy walk* in terms of the Markovian balance equations (4) proves to be very useful for the analysis of *Lévy walk* with a random death process [20]. Our theory can be useful to formulate the impact of the external force or the chemotactic substance on the random movement of particles with finite velocities. This theory can be also useful for the implementation of the non-linear reactions and development of the theory of wave propagation in reaction-transport systems involving enhanced diffusion and memory effects [16, 39].

Acknowledgement. This work was funded by EPSRC Grant No. EP/J019526/1 *Anomalous reaction-transport equations.* The author gratefully acknowledges the hospitality of the Isaac Newton Institute, Cambridge and thanks Nickolay Korabel and Vicenc Mendez for very

useful discussions during the programme *Coupling Geometric PDEs with Physics for Cell Morphology, Motility and Pattern Formation.* The author thanks Eli Barkai for useful suggestions.

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